KINCOPPAL-ROSE BAY

Cor Unum

2002 HALF YEARLY
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 120 minutes
- Write using black or blue pen
- Board-approved scientific calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 80

- Attempt all questions 1-4
- All questions are of equal value
- Start a new writing booklet for each question

Total marks – 80
Attempt Questions 1– 4
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (20 marks) Use a SEPARATE writing booklet.

(a) Find $\int_1^e (\ln x)^5 \frac{1}{x}\,dx$ $\frac{1}{6}$ **2**

(b) Find $\int \frac{1}{\sqrt{1-6x-x^2}}\,dx$ $\sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right)$ **3**

(c) Use integration by parts to evaluate $\int_0^{\frac{\pi}{3}} x\sec x\tan x\,dx$. **4**
Express your answer in the form $a + \log_e b$ where a and b are real. $\frac{2\pi}{3} + \ln(2-\sqrt{3})$

(d) (i) Decompose $\frac{8x+16}{(x-2)^2(x^2+4)}$ into partial fractions. **3**
$\frac{x-2}{x^2+4} - \frac{1}{x-2} + \frac{4}{(x-2)^2}$

(ii) Hence find $\int \frac{8x+16}{(x-2)^2(x^2+4)}\,dx$ $-\tan^{-1}\frac{x}{2} - \log(x-2) + \frac{1}{2}\log(x^2+4) - \frac{4}{x-2}$ **3**

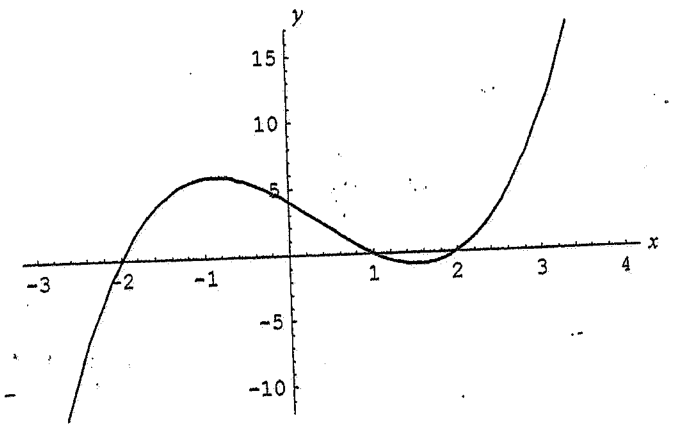
(e) (i) If $I_n = \int x^n e^x\,dx$, prove that $I_n = x^n e^x - nI_{n-1}$ **3**

(ii) Hence find $\int x^3 e^x\,dx$ $e^x(x^3-3x^2+6x-6)$ **2**

Marks

Question 2 (20 marks) Use a SEPARATE writing booklet.

(a) The curve $f(x)=(x+2)(x-1)(x-2)$ is sketched below. Trace the graph into your answer booklet.



(i) On the same number plane as the graph you have traced, sketch $y^2 = f(x)$ 3

(ii) On a separate number planes sketch the following for $-3 \le x \le 4$.

(α) $y=f(-x)$ 2

(β) $y=\dfrac{1}{f(x)}$ 3

(iii) On a separate number plane over the domain $0 \le x \le 2$ sketch $y=2^{f(x)}$ 3

(b) (i) Draw a neat sketch of $y=\sin^{-1}x$ showing all important features 2

(ii) On separate number planes draw neat sketches of:

(α) $y=\left(\sin^{-1}x\right)^2$ 2

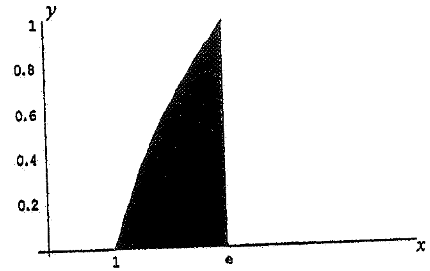
(β) $y=x\sin^{-1}x$ 3

(iii) Based solely on the evidence of your graph explain whether the function defined by $g(x)=x\sin^{-1}x$ is odd, even or neither. 2

Marks

Question 3 (20 marks) Use a SEPARATE writing booklet.

(a) The area enclosed by the curve $y=\ln x$ the x-axis and the line $x=e$,. is rotated about the line $x=e$.



(i) Copy the diagram into you answers and use it to help you draw a typical cylindrical shell. 1

(ii) Use the method of cylindrical shells to find the volume of the generated solid. 5

(b) The area enclosed by the curve $y=\ln x$ the x-axis and the line $x=e$,. is rotated about the line $x=e$.

(i) Draw a typical slice. 1

(ii) Find the volume of the slice. 2

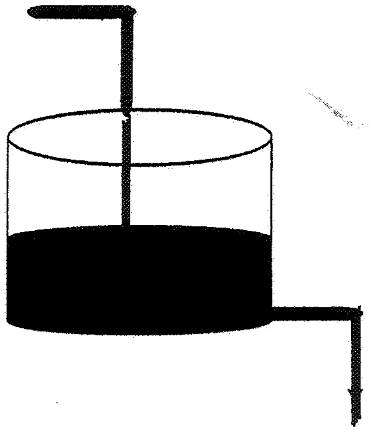
(iii) Find the volume of the solid using the method of slices. 4

(c) The Great Pyramid of Cheops has a square base of 230m in length. At the top it is also a square of length 10m. The height of the pyramid is 140m. Use the method of slices to determine the volume of the pyramid. 7

Marks

Question 4 (20 marks) Use a SEPARATE writing booklet.

(a)



In the diagram above, a tank initially contains 1000L of pure water. Salt water begins pouring into the tank from a pipe and an inbuilt stirrer ensures it is completely mixed with the pure water. A second pipe removes the mixture at the same rate, so there is always a total of 1000L in the tank.

(i) If the salt water entering the tank contains 2 grams of salt per litre, and is flowing in at the constant rate if w L/min, how much salt is entering the tank per minute? 1

(ii) If Q grams is the amount of salt in the tank at time t, how much salt is in 1 L at time t? 1

(iii) Hence write down the amount if salt leaving the tank per minute. 1

(iv) Use the previous parts to show that $\frac{dQ}{dt} = -\frac{w}{1000}(Q - 2000)$ 1

(v) Show that $Q = 2000 + Ae^{\frac{-wt}{1000}}$ is a solution of this differential equation. 1

(vi) Determine the value of A. 1

(vii) What happens to Q as $t \to \infty$? 1

(viii) If there is 1 kg of salt in the tank after $5\frac{3}{4}$ hours, find w. 1

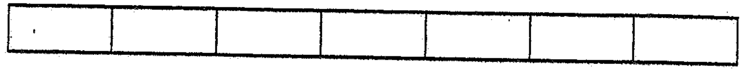
(b) You are reminded that a quadratic equation will have unreal roots if the discriminant is less than zero. Suppose that the coefficients b and c of the equation $x^2 + bx + c = 0$ are determined by rolling a die twice.

(i) Determine the probability the roots will be real. 2

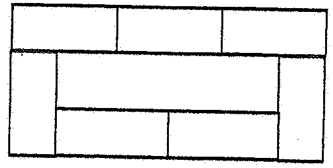
(ii) Given that the roots are real, determine the probability the roots are rational. 2

(c) Seven different coloured blocks are identical in size. The blocks are red, blue, green, white, yellow, violet and purple. Find the probability that red and green will not be adjacent to each other, if the blocks are arranged in the following patterns.

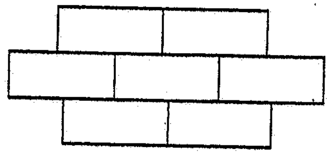
(i) 2



(ii) 3



(iii) 3



END OF PAPER

Total marks – 80
Attempt Questions 1– 4
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (20 marks) Use a SEPARATE writing booklet.

(a) let $u=\ln x, du=\frac{1}{x}dx, u_1=\ln e=1, u=\ln 1=0$ ☑

$\int_1^e (\ln x)^5 \frac{1}{x}dx=\int_0^1 u^5 du$

$=\left[\frac{u^6}{6}\right]_0^1$

$=\frac{1}{6}$ ☑

(b) $\int \frac{1}{\sqrt{1-6x-x^2}}dx=\int \frac{1}{\sqrt{10-(x^2+6x+9)}}dx$ ☑

$=\int \frac{1}{\sqrt{10-(x+3)^2}}dx$ ☑

$=\sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right)+C$ ☑

(c) $I=\int_0^{\frac{\pi}{3}} x\sec x\tan x\,dx \quad u=x \quad v=\sec x$

$u'=1 \quad v'=\sec x\tan x$ ☑

$I=[x\sec x]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\sec x\,dx$ ☑

$=\frac{\pi}{3}\times\sec\frac{\pi}{3}-[\ln(\sec x+\tan x)]_0^{\frac{\pi}{3}}$

$=\frac{2\pi}{3}-\left(\ln\left(\sqrt{2}+\sqrt{3}\right)-\ln 1\right)$

$=\frac{2\pi}{3}-\ln\left(\sqrt{2}+\sqrt{3}\right)$ ☑

$=\frac{2\pi}{3}+\ln\left(\sqrt{2}+\sqrt{3}\right)^{-1}$ ☑

$=\frac{2\pi}{3}+\ln\left(\sqrt{3}-\sqrt{2}\right)$

(d) (i) $\frac{8x+16}{(x-2)^2(x^2+4)}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{Cx+D}{(x^2+4)}$

$8x+16=A(x-2)(x^2+4)+B(x^2+4)+(Cx+D)(x-2)^2$

let $x=2$; $\quad 32=8B\rightarrow \underline{B=4}$

let $x=2i$; $\quad 16i+16=(2Ci+D)\times -8i$

let $x=2i$; $\quad 16i+16=16C+-8iD\rightarrow \underline{C=2}, \underline{D=-1}$

coeffs of x^3: $\quad 0=A+C\rightarrow \underline{A=-2}$ ☑ ☑ ☑

(ii) $\int\frac{8x+16}{(x-2)^2(x^2+4)}dx=\int\frac{-2}{(x-2)}+\frac{4}{(x-2)^2}+\frac{2x}{(x^2+4)}-\frac{1}{(x^2+4)}dx$

3

(e) (i) $I_n=\int x^n e^x dx$ integrating by parts with $u=x^n$ and $v'=e^x$ ☑

$I_n=x^n e^x-\int e^x n x^{n-1}dx$

$I_n=x^n e^x-n\int x^{n-1}e^x dx$ ☑

$I_n=x^n e^x-nI_{n-1}$ ☑

(ii) $\int x^3 e^x dx=I_3$

$I_3=x^3e^x-3I_2$

$=x^3e^x-3\left(x^2e^x-2I_1\right)$

$=x^3e^x-3x^2e^x+6I_1$

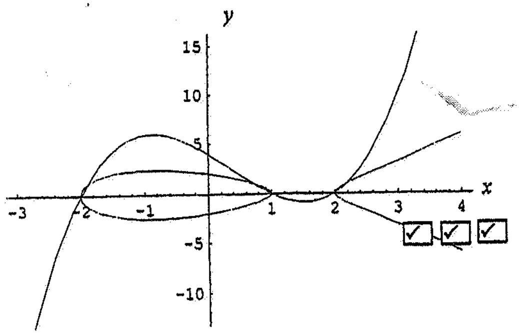
$=x^3e^x-3x^2e^x+6\left(xe^x-I_0\right)$

$=x^3e^x-3x^2e^x+6xe^x-6e^x+C$ ☑ ☑

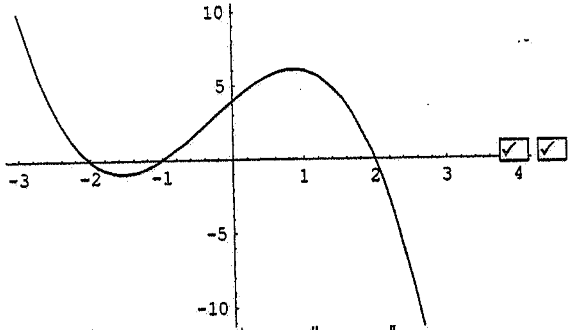
Marks

Question 2 (20 marks) Use a SEPARATE writing booklet.

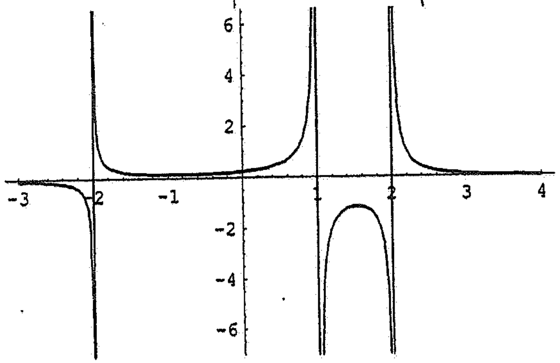
(a) (i)



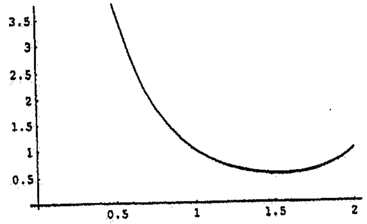
(ii) (α)



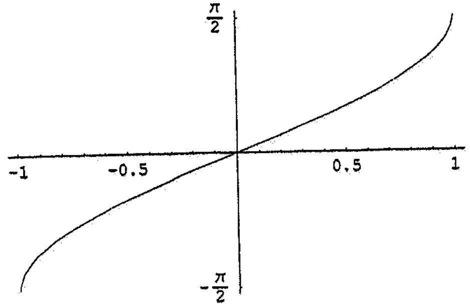
(β)



(iii)

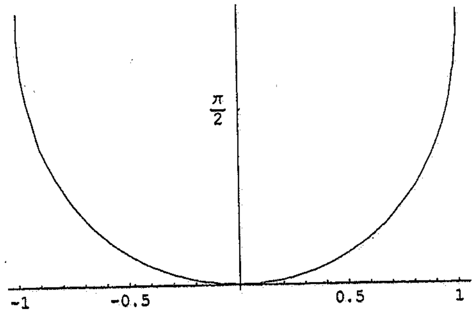


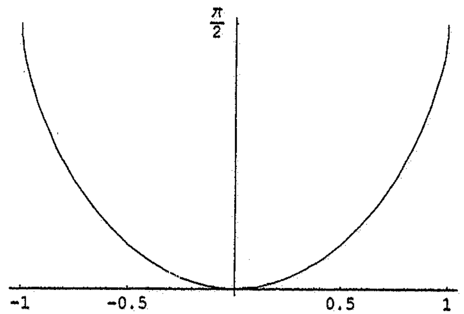
(b) (i) Draw a neat sketch of $y = \sin^{-1} x$ showing all important features **2**



(ii) On separate number planes draw neat sketches of:

(α)





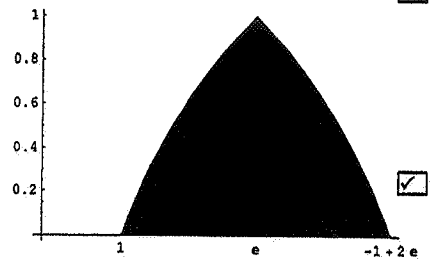
(β)

(i) $g(x) = x\sin^{-1}x$ is even sice it is reflected in the y-axis ✓ ✓

Marks

Question 3 (20 marks) Use a SEPARATE writing booklet.

(a) (i) (ii) ✓



✓

✓ ✓ ✓

$V(\text{shell}) = 2\pi \times \text{radius} \times \text{height} \times \text{thickness}$

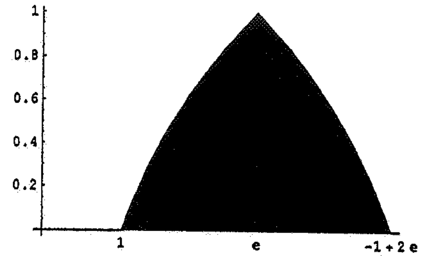
$V(\text{cylinder}) \approx \sum_{x=1}^{e} 2\pi \times (e - x) y \delta x$

$V(\text{cylinder}) = \lim_{\delta x \to 0} \sum_{x=1}^{e} 2\pi \times (e - x) y dx$

$V(\text{cylinder}) = \lim_{\delta x \to 0} \sum_{x=1}^{e} 2\pi \times (e - x) \log x dx$

$= 2\pi \int_1^e (e - x) \log x dx$

$= -\frac{\pi}{2}(e^2 - 4e + 1)$



$V(\text{slice}) = \pi (e - x)^2 \delta y$

$V \approx \sum_{y=0}^{1} \pi \times (e - x)^2 \delta y$ ✓ ✓

$V(\text{solid}) = \lim_{\delta y \to 0} \sum_{y=0}^{1} \pi \times (e - x)^2 \delta y$

$= \pi \int_0^1 (e - x)^2 dy$ ✓

$= \pi \int_0^1 (e - e^y)^2 dy$ ✓

$= -\frac{\pi}{2}(e^2 - 4e + 1)$ ✓ ✓

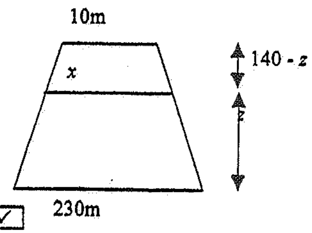
(c) Equating areas of trapezia:

$(x + 10)(140 - z) + (x + 230)z = 240 \times 140$ ✓

$x = 230 - \frac{11z}{7}$ ✓ ✓

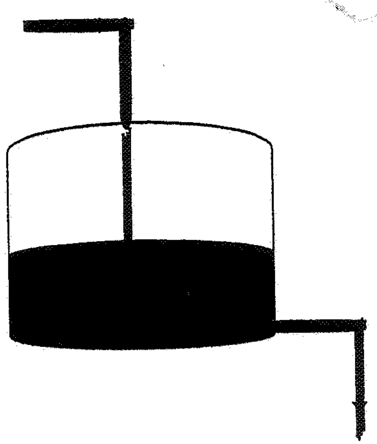
$V = \int_0^{140} \left(230 - \frac{11z}{7}\right)^2 dz$ ✓

$= 7742000/3u^3$ ✓ ✓ ✓



Question 4 (20 marks) Use a SEPARATE writing booklet.

(a)



In the diagram above, a tank initially contains 1000L of pure water. Salt water begins pouring into the tank from a pipe and an inbuilt stirrer ensures it is completely mixed with the pure water. A second pipe removes the mixture at the same rate, so there is always a total of 1000L in the tank.

(i) $2w$ L/min ☑

(ii) If Q /1000L ☑

(iii) $2wQ$ /1000L = $wQ/500$ ☑

(iv) Use the previous parts to show that $\frac{dQ}{dt} = -\frac{w}{1000}(Q-2000)$ ☑

(v) $Q = 2000 + Ae^{\frac{-wt}{1000}}$ ☑

$\frac{dQ}{dt} = \frac{-w}{1000}Ae^{\frac{-wt}{1000}}$

$\frac{dQ}{dt} = \frac{-w}{1000}(Q-2000)$ ☑

(vi) $A = -1000$. ☑

(vii) $Q \to 2000$?

(viii) If there is 1 kg of salt in the tank after $5\frac{3}{4}$ hours, find w. 1

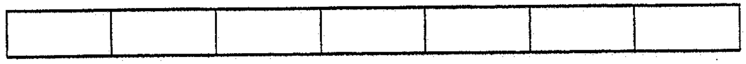
(b) Real roots if $b^2 - 4c \geq 0 \to b^2 \geq 4c$:(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)

$P(real\ roots) = \frac{19}{36}$ ☑ ☑

(ii) Rational roots if $b^2 - 4c$ is a perfect square. (4,4), (5,5), (5,6) ,(6,6) $P(rationals) = \frac{4}{19}$ ☑ ☑

(c) Seven different coloured blocks are identical in size. The blocks are red, blue, green, white, yellow, violet and purple. Find the probability that red and green will not be adjacent to each other, if the blocks are arranged in the following patterns.

(i) 2



$P(\text{not adjacent}) = 1 - P(\text{adjacent})$

$= 1 - \left(\frac{6! \times 2!}{7!}\right)$

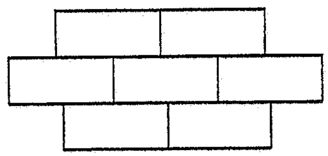
$= \frac{5}{7}$

(ii) $P(\text{not adjacent}) = 1 - P(\text{adjacent})$

$= 1 - \left(\frac{5! \times 2!}{6!}\right)$

$= \frac{2}{3}$

(iii)



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$P(\text{not adjacent}) = \left(\frac{4\times 3\times 5! + 2\times 3\times 5!}{7!}\right)$

$= \frac{3}{7}$

END OF PAPER